APPLICATION OF THE L'HÔPITAL'S RULE

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https://doi.org/10.5281/zenodo.10969560

Abstract. *This article studies the main types of uncertainties that arise when finding limits. Some methods of disclosure of uncertainties are given.*

Key words: *Taylor series, limit point, L'Hôpital's rules, logarithm, exponential.*

ПРИМЕНЕНИЕ ПРАВИЛА ЛОПИТАЛЯ

Аннотация. *В данной статье изучаются основные виды неопределенностей, возникающих при нахождении пределов. Приведены некоторые методы раскрытия неопределённостей.*

Ключевые слова: *Ряд Тейлора, предельная точка, правила Лопиталя, логарифм, экспонент.*

INTRODUCTION

Uncertainty disclosure – methods for calculating the limits of a function given by formulas, which, as a result of a formal substitution of an argument in them, lose their meaning, that is, they turn into expressions like: $(\infty-\infty)$, (∞/∞), $(0/0)$, $(0\cdot\infty)$, (0^0), (1^∞), (∞^0). Here, 0 is an infinitesimal value, and ∞ is an infinitely large value.

Revealing uncertainties allows:

1. Simplification of the type of function (transformation using abbreviated multiplication formulas, trigonometric formulas, etc.).
2. Use of wonderful limits.
3. Application of L'Hôpital's rule.
4. Using the replacement of an infinitesimal expression with its equivalent.

OBJECT AND METHODS OF THE RESEARCH

The most powerful method is L'Hôpital's rule, however, it does not allow calculating the limit in all cases. A method for this kind of uncertainty was published in the 1696 textbook "Analyse des Infiniment Petits" by Guillaume Lopital. The method was communicated to Lopital in a letter by its discoverer Johann Bernoulli. In addition, it is directly applicable only to the second and third of the listed types of uncertainties, that is, the relation, and in order to reveal other types, they must first be reduced to one of these. Also, to calculate the limits, the expansion of the expressions included in the uncertainty under study is often used in a Taylor series in the vicinity of the limit point.

RESULTS AND THEIR DISCUSSION

L'Hôpital's first rule (uncertainty of the form $0/0$ at $x\to a-$).

Theorem 1. Let:

1) $f(x)$ and $g(x)$ are defined and differentiable in some interval $(a-b_1,a)$, $b_1>0$;

2) $\lim\limits_{x\to a-} f(x)=\lim\limits_{x\to a-} g(x)=0$;

3) $f'(x), g'(x) \neq 0$ for all x belonging to $(a-b_2, a)$ for some $b_2 > 0$;

4) there is a finite or infinite limit at $x \to a-$ of the relation $\frac{f'(x)}{g'(x)}$, i.e. $\lim_{x\to a-}\frac{f'(x)}{g'(x)}$.

Then there is a limit of the relation $\frac{f(x)}{g(x)}$ and the equality

$$\lim_{x\to a-}\frac{f(x)}{g(x)} = \lim_{x\to a-}\frac{f'(x)}{g'(x)}.$$

The second L'Hôpital's rule (uncertainty of the form ∞/∞ for $x \to a$).

Theorem 2. Let the functions $f(x)$ and $g(x)$ be defined and differentiable in the interval (a,b); $g'(x) \neq 0$ for all x belonging to (a,b); $f(x) \to \infty$ and $g(x) \to \infty$ at $x \to a$; there is a finite or infinite limit of $\lim_{x\to a}\frac{f'(x)}{g'(x)}$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

To reveal the uncertainties (0^0), (1^∞), (∞^0) we use the following method: we find the limit of the (natural) logarithm of the expression containing the given uncertainty. As a result, the type of uncertainty changes. After finding the limit, we take the exponent from it.

$$\left(0^0\right) = \left(e^{0\cdot\ln 0}\right) = \left(e^{0\cdot(-\infty)}\right);\ \left(1^\infty\right) = \left(e^{\infty\cdot\ln 1}\right) = \left(e^{\infty\cdot 0}\right);\ \left(\infty^0\right) = \left(e^{0\cdot\ln(\infty)}\right) = \left(e^{0\cdot(\infty)}\right).$$

Example 1. Is it possible to apply L'Hôpital's rule to the limit

$$\lim_{x\to 0}\frac{x^2\cdot\sin\frac{1}{x}}{\sin x}$$

The functions $f(x) = x^2\cdot\sin\frac{1}{x}$ and $g(x) = \sin x$, $x \in R\setminus\{0\}$ are defined and continuous in a neighborhood of the point $x = 0$ (excluding the point $x = 0$); their derivatives

$$f'(x) = 2x\cdot\sin\frac{1}{x} - \cos\frac{1}{x} \text{ and } g'(x) = \cos x$$

simultaneously exist for $x \neq 0$; their expression

$$\left(f'(x)\right)^2 + \left(g'(x)\right)^2 = \cos^2 x + \cos^2\frac{1}{x} - 2x\sin\frac{2}{x} + 4x^2\sin^2\frac{1}{x} \text{ at } x \neq 0 \text{ and}$$

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{2x\cdot\sin\frac{1}{x} - \cos\frac{1}{x}}{\cos x} \quad (1)$$

Since $\lim_{x\to0}\left(2x\cdot\sin\frac{1}{x}\right)\cdot(\cos x)^{-1}=0$ and $\lim_{x\to0}\left(\cos\frac{1}{x}\right)\cdot(\cos x)^{-1}$ does not exist, the limit (1) also does not exist. Therefore, the application of L'Hôpital's rule in this example is impossible. Note that

$$\lim_{x\to0}\frac{x}{\sin x}\cdot x\cdot\sin\frac{1}{x}=0.$$

Example 2. Find the limit $w=\lim_{x\to0}\left(\frac{1+e^x}{2}\right)^{cthx}$.

Uncertainty is brought to the form $e^{\frac{0}{0}}$, we get

$$\left(\frac{1+e^x}{2}\right)^{cthx}=e^{\ln\left(\frac{1+e^x}{2}\right)\cdot(thx)^{-1}}$$

and applying L'Hôpital's rule we have

$$\lim_{x\to0}\frac{\ln\left(\frac{1+e^x}{2}\right)}{thx}=\lim_{x\to0}\frac{\frac{2}{1+e^x}\cdot\frac{e^x}{2}}{ch^{-2}x}=\frac{1}{2}.$$

Thus, $w=e^{\frac{1}{2}}=\sqrt{e}$.

Example 3. Find the limit $\lim_{x\to1}\frac{x^{x+1}(\ln x+1)-x}{1-x}$.

Function $f(x)=x^{x+1}(\ln x+1)-x$ and $g(x)=1-x$, $x>0$, $x\neq1$ satisfy the following conditions:

1) $\lim_{x\to1}f(x)=\lim_{x\to1}g(x)=0$;

2) their derivatives $f'(x)=x^{x+1}(\ln x+1)\left(1+\frac{1}{x}+\ln x\right)+x^x-1$, $g'(x)=-1$, exist for $x>0$;

3) $\lim_{x\to1}\frac{f'(x)}{g'(x)}=-2$ exists;

4) $(f'(x))^2+(g'(x))^2\neq0$ at $x>0$.

Therefore, the first L'Hôpital's rule applies, according to which we have

$$\lim_{x\to1}\frac{x^{x+1}(\ln x+1)-x}{1-x}=\lim_{x\to1}\frac{f'(x)}{g'(x)}=-2$$

Example 4. Find the limit $\lim_{x\to1}\frac{x^x-x}{\ln x-x+1}$.

The function $f(x)=x^x-x$ and $g(x)=\ln x-x+1$, $x>0$, $x\neq 1$ satisfy the following conditions:

1) $\lim\limits_{x\to 1} f(x)=\lim\limits_{x\to 1} g(x)=0$.

2) derivatives $f'(x)=x^x(\ln x+1)-1$, $g'(x)=\dfrac{1}{x}-1$ exists in a sufficiently small neighborhood of the point $x=1$.

3) for in the indicated neighborhood.

4) according to the previous example, there is a finite limit.

$$\lim_{x\to 1}\frac{f'(x)}{g'(x)}=\lim_{x\to 1}\frac{x^{x+1}(\ln x+1)-x}{1-x}=-2.$$

Therefore, the first rule of L'Hôpital applies, and we have

$$\lim_{x\to 1}\frac{f(x)}{g(x)}=\lim_{x\to 1}\frac{x^{x+1}(\ln x+1)-x}{1-x}=-2.$$

Example 5. Find the limit of a matrix function

$$A(x)=\begin{pmatrix}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}} & \left(\dfrac{arctgx}{x}\right)^{\frac{1}{x^2}}\\ \left(\dfrac{Arshx}{x}\right)^{\frac{1}{x^2}} & \left(\dfrac{(1+x)^{\frac{1}{x}}}{e}\right)^{\frac{1}{x}}\end{pmatrix}.$$

Since $\lim\limits_{x\to a} A(x)=\left(\lim\limits_{x\to a} a_{ij}(x)\right)$, where are $a_{ij}(x)$ the elements of the functional matrix $A(x)$, then we calculate the limit of this matrix element by element. We have

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}=e^z, \text{ where } z=\lim_{x\to 0}\frac{\ln\frac{\sin x}{x}}{x^2}.$$

Applying L'Hôpital's rule

$$z=\lim_{x\to 0}\frac{x}{2x\sin x}\cdot\frac{x\cos x-\sin x}{x^2}=\lim_{x\to 0}\frac{x\cos x-\sin x}{2x^3}=\lim_{x\to 0}\frac{x\sin x}{6x^2}=-\frac{1}{6}.$$

Similarly, we obtain for all other elements:

$$\lim_{x\to 0}\left(\frac{arctgx}{x}\right)^{\frac{1}{x^2}}=e^z,$$

$$z=\lim_{x\to 0}\frac{\ln\frac{arctgx}{x}}{x^2}=\lim_{x\to 0}\frac{x}{arctgx}\cdot\frac{\frac{x}{1+x^2}-arctgx}{2x^3}=\lim_{x\to 0}\frac{x-(1+x^2)arctgx}{2x^3}=$$

$$=\lim_{x\to 0}\frac{x-(1+x^2)arctgx}{2x^3}=\lim_{x\to 0}\frac{2x\cdot arctgx}{6x^2}=-\frac{1}{3}.$$

$$\lim_{x\to 0}\left(\frac{Arshx}{x}\right)^{\frac{1}{x^2}}=e^z,$$

$$z=\lim_{x\to 0}\frac{\ln\frac{Arshx}{x}}{x^2}=\lim_{x\to 0}\frac{Arshx}{x}\cdot\frac{\frac{x}{\sqrt{1+x^2}}-Arshx}{2x^3}=\frac{1}{2}\lim_{x\to 0}\frac{x-u(x)}{x^3}=$$

$$=\lim_{x\to 0}\frac{Arshx}{3x^2}=-\frac{1}{6}.$$

(Here we have introduced the notation $u(x)=\sqrt{1+x^2}\cdot Arshx$.)

$$\lim_{x\to 0}\left(\frac{(1+x)^{\frac{1}{x}}}{e}\right)^{\frac{1}{x}}=e^z,$$

$$z=\lim_{x\to 0}\frac{\ln\frac{(1+x)^{\frac{1}{x}}}{e}}{x}=\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2}=\lim_{x\to 0}\frac{\frac{1}{1+x}-1}{2x}=\lim_{x\to 0}\frac{-\frac{1}{(1+x^2)}}{2}=\frac{1}{2}.$$

So, finally we have $\lim_{x\to 0}A(x)=\begin{pmatrix} e^{-\frac{1}{6}} & e^{-\frac{1}{3}} \\ e^{-\frac{1}{6}} & e^{-\frac{1}{2}} \end{pmatrix}$.

Example 6. Find $z=\lim_{x\to 0}\frac{\det\begin{pmatrix} x & \sin x \\ e^x-1 & 1+x^2 \end{pmatrix}}{\det\begin{pmatrix} x\cos x & tgx \\ chx & e^x \end{pmatrix}}$.

These determinants as functions of a variable satisfy all the conditions of L'Hôspital's rule in some neighborhood of the point $x=0$. Therefore, applying the rule, we get

$$z=\lim_{x\to 0}\frac{\det\begin{pmatrix} x & \sin x \\ e^x-1 & 1+x^2 \end{pmatrix}}{\det\begin{pmatrix} x\cos x & tgx \\ chx & e^x \end{pmatrix}}=\lim_{x\to 0}\frac{\det\begin{pmatrix} 1 & \cos x \\ e^x-1 & 1+x^2 \end{pmatrix}+\det\begin{pmatrix} x & \sin x \\ e^x & 2x \end{pmatrix}}{\det\begin{pmatrix} \cos x-x\sin x & \cos^{-2}x \\ chx & e^x \end{pmatrix}+\det\begin{pmatrix} x\cos x & tgx \\ shx & e^x \end{pmatrix}}=$$

$$=\lim_{x\to 0}\frac{\det\begin{pmatrix}1&1\\0&1\end{pmatrix}+\det\begin{pmatrix}0&0\\1&0\end{pmatrix}}{\det\begin{pmatrix}1&1\\0&1\end{pmatrix}+\det\begin{pmatrix}0&0\\1&1\end{pmatrix}}=1.$$

Example 7. Find the asymptote of the curve $y=\frac{x^{1+x}}{(1+x)^x}$, $x>0$.

The oblique asymptote equation has the form $y=kx+b$. Using the equation of the curve, we find k and b:

$$k=\lim_{x\to+\infty}\frac{x^x}{(1+x)^x}=\lim_{x\to+\infty}\frac{1}{\left(1+\frac{1}{x}\right)^x}=\frac{1}{e},$$

$$b=\lim_{x\to+\infty}\left(\frac{x^{1+x}}{(1+x)^x}-\frac{x}{e}\right)=\lim_{x\to+\infty}x\cdot\left(\frac{1}{\left(1+\frac{1}{x}\right)^x}-\frac{1}{e}\right)=\frac{1}{e^2}\cdot\lim_{x\to+\infty}x\cdot\left(e-\left(1+\frac{1}{x}\right)^x\right)=$$

$$=\frac{1}{e^2}\cdot\lim_{t\to+0}\left(\frac{e-(1+t)^{\frac{1}{t}}}{t}\right)=\frac{1}{e^2}\cdot\lim_{t\to+0}(1+t)^{\frac{1}{t}}\cdot\left(\frac{1}{t(t+1)}-\frac{\ln(1+t)}{t^2}\right)=$$

$$=-\frac{1}{e^2}\cdot\lim_{t\to+0}\frac{t-(1+t)\ln(1+t)}{t^2(t+1)}=-\frac{1}{e^2}\cdot\lim_{t\to+0}\frac{\ln(1+t)}{2t+3t^2}=\frac{1}{2e}.$$

Thus, we obtain the asymptote equation $y=\frac{x}{e}+\frac{1}{2e}$.

Example 8. Investigate for differentiability at a point at point $x=0$ the function

$$f(x)=\begin{cases}\frac{1}{x}-\frac{1}{e^x-1}, & \text{if } x\neq 0\\ \frac{1}{2}, & \text{if } x=0\end{cases}.$$

To investigate the differentiability of a function at a point $x=0$ means to establish the existence of a finite limit

$$f'(0)=\lim_{x\to 0}\frac{\frac{1}{x}-\frac{1}{e^x-1}-\frac{1}{2}}{x}\qquad(1).$$

We will search for the limit of (1) according to L'Hôpital's rule, for which we must make sure that the numerator in (1) tends to zero as $x\to 0$. A test using L'Hôspital's rule shows that

ISSN: 2181-3906
2024

International scientific journal
«MODERN SCIENCE AND RESEARCH»
VOLUME 3 / ISSUE 4 / UIF:8.2 / MODERNSCIENCE.UZ

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}-\frac{1}{2}\right)=\lim_{x\to 0}\frac{2e^x-2-x-xe^x}{2x\left(e^x-1\right)}=\lim_{x\to 0}\frac{e^x-1-xe^x}{2\left(e^x(x+1)-1\right)}=\frac{1}{2}\lim_{x\to 0}\frac{-xe^x}{2e^x+xe^x}=0$$

.

So, in the formula (1) we have an uncertainty of the form $\frac{0}{0}$. Applying the L'Hôpital rule to (1) three times, we get

$$\lim_{x\to 0}\frac{\frac{1}{x}-\frac{1}{e^x-1}-\frac{1}{2}}{x}=\lim_{x\to 0}\frac{2e^x-2-x-xe^x}{2x^2\left(e^x-1\right)}=\lim_{x\to 0}\frac{2e^x-1-e^x-xe^x}{4x\left(e^x-1\right)+2x^2e^x}=$$

$$=\lim_{x\to 0}\frac{-xe^x}{4x\left(e^x-1\right)+e^x\left(8x+2x^2\right)}=\lim_{x\to 0}\frac{-e^x(x+1)}{\left(12+12x+2x^2\right)e^x}=-\frac{1}{12},$$

$$f'(0)=-\frac{1}{12}.$$

Example 9. $w=\lim_{x\to+\infty}\left(\left(tg\frac{\pi x}{2x+1}\right)^{\frac{1}{x}},\left(\frac{2}{\pi}arctgx\right)^{x},(thx)^{x}\right).$

To find the limit of a vector function, we calculate the limits of each of its components. Since the components are exponential expressions, we apply the representation $u^v=e^{v\ln u}$, $u>0$, and, having reduced the corresponding uncertainties to the form $\frac{0}{0}$, we use the L'Hôpital rule. We have

$$\lim_{x\to+\infty}\left(tg\frac{\pi x}{2x+1}\right)^{\frac{1}{x}}=e^{\lim\limits_{x\to+\infty}\left(tg\frac{\pi x}{2x+1}\right)^{-1}\cdot\cos^{-2}\frac{\pi x}{2x+1}\cdot\frac{\pi}{(2x+1)^2}}=e^{2\pi\cdot\lim\limits_{x\to+\infty}\left(\sin\frac{2\pi x}{2x+1}\right)^{-1}\cdot(2x+1)^{-2}}=e^{2\cdot\lim\limits_{x\to\infty}\frac{\frac{\pi x}{2x+1}}{\sin\frac{\pi x}{2x+1}}\cdot\alpha}=1,$$

$$\alpha=\lim_{x\to\infty}\frac{1}{2x+1}=0,$$

$$\lim_{x\to+\infty}\left(\frac{2}{\pi}arctgx\right)^{x}=e^{\lim\limits_{x\to+\infty}x\cdot\ln\left(\frac{2}{\pi}arctgx\right)}=e^{z}\text{, где } z=\lim_{x\to+\infty}\frac{\frac{1}{1+x^2}\cdot\frac{1}{arctgx}}{-\frac{1}{x^2}}=-\frac{2}{\pi},$$

$$\lim_{x\to+\infty}(thx)^{x}=e^{\lim\limits_{x\to+\infty}x\cdot\ln(thx)}=e^{z},$$

ISSN: 2181-3906 2024

International scientific journal «MODERN SCIENCE AND RESEARCH»

VOLUME 3 / ISSUE 4 / UIF:8.2 / MODERNSCIENCE.UZ

$$z=\lim_{x\to+\infty} x\ln(thx)=\lim_{x\to+\infty}\frac{\frac{1}{thx}\cdot\frac{1}{ch^2x}}{-\frac{1}{x^2}}=-2\cdot\lim_{x\to+\infty}\frac{x^2}{shx}=-2\cdot\lim_{x\to+\infty}\frac{x}{chx}=0.$$

Consequently, $w=\left(1,e^{-\frac{2}{\pi}},1\right)$.

Example 10. $w=\lim_{x\to+0}\left(e^{-\frac{1}{x^2}}\cdot x^{-100},x^{x^x-1}\right)$,

Since for vector - functions

$$w=\left(e^{-\frac{1}{x^2}}\cdot x^{-100},x^{x^x-1}\right)=\left(\lim_{x\to+0}\left(e^{-\frac{1}{x^2}}\cdot x^{-100}\right),\lim_{x\to+0}\left(x^{x^x-1}\right)\right),$$

then we find the limits of each of the components separately. We have

$$\lim_{x\to+0}\left(e^{-\frac{1}{x^2}}\cdot x^{-100}\right)=\lim_{y\to+\infty}\frac{y^{50}}{e^y}=50!\lim_{y\to+\infty}e^{-y}=0.$$

(here the second L'LHôspital's rule is applied 50 times).

For the second component, we first apply the representation $u^v=e^{v\ln u}$, $u>0$, and perform some transformation so that we can use the L'Hôpital rule:

$$\lim_{x\to+0}\left(x^{x^x-1}\right)=\lim_{x\to+0}e^{x\ln^2 x\left(\frac{e^{x\ln x}-1}{x\ln x}\right)}=e^{ab}.$$

(Here we have used the continuity of the function $f(x)=e^x$ and the product limit theorem). To find $a=\lim_{x\to+0}x\ln^2 x=\lim_{x\to+0}\frac{\ln^2 x}{x^{-1}}$, we use the second, and to find $b=\lim_{x\to+0}\frac{e^{x\ln x}-1}{x\ln x}$, the first rule of L'Hôpital. We have

$$a=\lim_{x\to+0}\frac{\ln^2 x}{x^{-1}}=\lim_{x\to+0}\frac{2\cdot\ln x\cdot\frac{1}{x}}{-x^{-2}}=\lim_{x\to+0}\frac{2\cdot\ln x}{-x^{-1}}=\lim_{x\to+0}\frac{-\frac{2}{x}}{-\frac{1}{x^2}}=0.$$

$$b=\lim_{x\to+0}\frac{e^{x\ln x}-1}{x\ln x}=\lim_{t\to-0}\frac{e^t-1}{t}=1.$$

Therefore, finally $w=(0,1)$.

CONCLUSION

L'Hôpital's rule often (though not always) allows you to reveal uncertainties of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ without much thought - if after the first differentiation you again get uncertainty, it does not matter - you can differentiate again, and so on until we get some specific limit.

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